IDENTIFIABILITY, IN THE LARGE, OF A NONLINEAR HEAT-CONDUCTION EQUATION AND SELF-SIMILAR SOLUTIONS

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Conditions are considered for the spoilage of the mutually one-to-one correspondence between the temperature field of a plate and its thermophysical properties that depend on the temperature.

The so-called inverse problems attract more and more attention in the theory of heat conduction. They are characteristic of the identification of a given differential equation by accessible observations on the properties of the object under investigation whose mathematical model is the equation under consideration.

The solutions of such problems touch upon questions of the spoilage of the correctness of the mathematical formulation and, in particular, require the build-up of the uniqueness of determining the reason by the observable consequence. A study of this question for problems to find the volume specific heat and the heat conductivity simultaneously, that depend on the temperature, was carried out in [1, 2]. In the former it is proved that among the temperature fields of a plate there is that by which it is impossible to restore its thermophysical properties uniquely. At the same time, in the second is shown the uniqueness of the solution for the designated inverse problem upon giving boundary conditions of the second kind. The results obtained indicated that different experiment conditions exist, depending on which, unique or nonunique values of the desired coefficients can be found.

A further study is made in this paper of the conditions that cause ambiguity in the restoration of the thermophysical properties by observations on a single temperature field. This question is investigated from the viewpoint of the spoilage of the mutually one-to-one correspondence between the solution of the boundary-value problem and its coefficients [3]. Such a formulation of the question of the identifiability of the mathematical model permits finding the ambiguity subset, containing a family of dissimilar thermophysical properties to which the same temperature field will correspond. The approach developed differs from those known [4, 5] in which the main attention is paid to extracting a certain uniqueness class in a set of initial data. Investigation of the opposite question, i.e., determination of the conditions for the existence of nonunique desired quantities, is directed at the extraction of a nonempty subset of ambiguity among the set of coefficients of the mathematical model. Then conditions for the existence of unique desired quantities follow from the requirement of emptiness of this subset, and the appropriate model states belong to the uniqueness class. In this connection, in solving the inverse problems it is sufficient to show that the observations obtained do not belong to the subspace of nonidentifiable states. A study of the mutually one-to-one correspondence permits obtaining simple criteria for this [3] by which the form of the nonidentifiable temperature fields as well as the properties of the external effects and the boundary conditions can be expressed. In this plan, the investigation of the question of the nonuniqueness of the solution of the inverse problem can be a basis for obtaining more general conditions for the identifiability of the mathematical model.

In the domain $Q_T = \{(x, t): x_1 < x < x_2,\ 0 < t < T\}$ we consider a one-dimensional nonlinear equation of heat conductivity with boundary conditions of the first kind

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$$a_1 \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a_2 \frac{\partial u}{\partial x} \right), \ (x,\ t) \in Q_T,$$
$$u|_{t=0} = \varphi(x),\ x_1 < x < x_2, \tag{1}$$
$$u|_{x=x_1} = \gamma_1(t),\ u|_{x=x_2} = \gamma_2(t),\ 0 < t < T,$$

where $a_1(u)$ and $a_2(u)$ are positive, or continuous and differentiable, functions. With respect to the boundary conditions $\Phi(x)$ and $\gamma_{1,2}(t)$ it is assumed that they satisfy the conditions [6] assuring the existence and uniqueness of the solution $u(x, t) \in C^{2,1}(\bar{Q}_T)$ for the given coefficients $a_{1,2}(u)$. We shall also consider the transformations executed below on the coefficients to be satisfied to the accuracy of a constant.

Let us show that in this case the solution of problem (1), which is not identifiable in the large, has the form $u_* = u_*(\xi)$ and satisfies the equation

$$\frac{du_*}{d\xi} = \exp\left[\int_0^{u_*} \lambda(v)\, dv \right], \tag{2}$$

and the dissimilarity subset is expressed to the accuracy of a constant by the family

$$a_1 = \exp\left[\int_0^{u} \lambda(v)\, dv \right] \frac{\xi_x^2}{\xi_t} \left(\lambda a_2 + \frac{da_2}{du} \right), \tag{3}$$

where

$$\xi = ax + bt + c \tag{4}$$

or

$$\xi = \frac{a + x}{\sqrt{b(c - t)}} + d, \tag{5}$$

and a, b, c, d = const.

We assume for the proof that there exist vector functions $\mathbf{a}' \neq \mathbf{a}''$, where $\mathbf{a} = \{a_1(u), a^2(u)\}$, to which the identical solution $u_*(x, t) \in C^{2,1}(\bar{Q}_T)$ corresponds, where a_i'/a_i'' = const, i = 1, 2. Subtracting the heat-conduction equations with these values of the coefficients from each other, we obtain

$$k_1(u) \frac{\partial u_*}{\partial t} = k_2(u) \frac{\partial^2 u_*}{\partial x^2} + \frac{dk_2}{du} \left(\frac{\partial u_*}{\partial x} \right)^2, \tag{6}$$

where $k_1 = a_1' - a_1''$, $k_2 = a_2' - a_2''$. Determining the highest derivative $\partial^2 u_*/\partial x^2$ from (6) and substituting it into the heat-conduction equation, we find

$$\rho(u) \frac{\partial u_*}{\partial t} = \left(\frac{\partial u_*}{\partial x} \right)^2, \tag{7}$$

where

$$\rho(u) = (a_1 k_2 - a_2 k_1) \Big/ \left(k_2 \frac{da_2}{du} - a_2 \frac{dk_2}{du} \right). \tag{8}$$

Substituting the expression for the derivative $(\partial u_*/\partial x)^2$ from (7) into (6), we obtain

$$\left(k_1 - \rho \frac{dk_2}{du} \right) \frac{\partial u_*}{\partial t} = k_2 \frac{\partial^2 u_*}{\partial x^2}. \tag{9}$$

It follows from (7) and (9) that

$$\lambda(u) \left(\frac{\partial u_*}{\partial x} \right)^2 = \frac{\partial^2 u_*}{\partial x^2}, \tag{10}$$

where $\lambda(u) = \left(k_1 - \rho \frac{dk_2}{du}\right) \Big/ k_2\rho$.

By using the functions $\rho(u)$ and $\lambda(u)$, expression (8) permits finding the family of coefficients

$$a_1 = \rho\left(\lambda a_2 + \frac{da_2}{du}\right), \tag{11}$$

to which the very same solution u_* of problem (1) corresponds. Analogously to [1], we make the following substitution in (10)

$$\frac{d\xi}{du_*} = \exp\left[-\int_0^{u_*} \lambda(v)\, dv\right]. \tag{12}$$

It permits obtaining $-\frac{d\xi}{du_*}\frac{\partial}{\partial x}(\xi_x^{-1}) + \lambda = \lambda$. It hence follows that the function $\xi(x, t)$ should have the form

$$\xi = \alpha(t)x + \beta(t). \tag{13}$$

To determine the functions $\alpha(t)$ and $\beta(t)$, we consider (7) with (12) taken into account, i.e., $u_* = u_*(\xi)$. Then

$$\rho = \frac{du_*}{d\xi}\frac{\xi_x^2}{\xi_t}. \tag{14}$$

Hence $\xi_t/\xi_x^2 = \frac{du_*}{d\xi}\Big/\rho$. Since the right side of the last expression is a function of the variable ξ according to (8), then we obtain $\xi_t/\xi_x^2 = F(\xi)$. On the other hand, the ratio between the derivatives of the function $\xi(x, t)$ is determined from (13) and has the form $\xi_t/\xi_x^2 = \left(x\frac{d\alpha}{dt} + \frac{d\beta}{dt}\right)\Big/\alpha^2$. From a comparison of the last two expressions there follows that the function $F(\xi)$ should be linear, and hence

$$\left(x\frac{d\alpha}{dt} + \frac{d\beta}{dt}\right)\Big/\alpha^2 = C_1(\alpha x + \beta) + C_2,$$

where $C_{1,2}$ are arbitrary constants. This latter relationship results in the system

$$\frac{d\alpha}{dt} = C_1\alpha^3, \quad \frac{d\beta}{dt} = (C_1\beta + C_2)\alpha^2.$$

Its solution $\alpha = a$, $\beta = bt + c$, if $C_1 = 0$, and otherwise

$$\alpha = \frac{1}{\sqrt{b(c-t)}}, \quad \beta = \frac{a}{\sqrt{b(c-t)}} + d,$$

where a, b, c, d = const. We hence obtain that the function $\xi(x, t)$ should have the form (4) or (5). As follows from (12), with respect to it the solution of the problem (1) will satisfy (2).

For given values of the coefficients $a_{1,2}(u)$ the form of the function $\lambda(u)$ is found from (11) and (14). It is the solution of the equation

$$\frac{d\lambda}{du} + \lambda^2 + \lambda\frac{d}{du}\ln\left(\frac{\xi_x^2}{\xi_t}\frac{a_2^2}{a_1}\right) = \frac{1}{a_2}\frac{da_2}{du}\frac{d}{du}\ln\left(\frac{\xi_t}{\xi_x^2}\frac{a_1}{\frac{da_2}{du}}\right). \tag{15}$$

Therefore, it is proved that within the framework of formulation (1) there exists a family of coefficients (3) to which the very same solution of the first boundary-value problem for the nonlinear heat-conduction equation corresponds. Their sufficiency to spoil the mutually one-to-one correspondence between the temperature field and the thermophysical properties of the plate is established by direct substitution of a function of the type (2) and a family (3) in the equation under consideration. Therefore, spoilage of the mutually one-to-one correspondence occurs in the above-mentioned sense if and only if the temperature field is expressed by a function of the type (2), where the parameter $\lambda(u)$ is found for given $a_{1,2}(u)$ from the solution of (15).

From the viewpoint of the invariant properties of differential equations [7], family (3) found exhibits the form of the transformation of the thermophysical properties for which the temperature field is conserved in the whole domain of variation of the independent variables of the initial equation. The family (3) permits finding the following transport group:

$$a_1'' = a_1' + \exp\left[\int\limits_0^u \lambda(v)\,dv\right]\frac{\xi_x^2}{\xi_t}\left(\lambda h + \frac{dh}{du}\right),\quad a_2'' = a_2' + h, \tag{16}$$

where h is a given change in the heat-conduction coefficient.

Let us turn to the question of the uniqueness of the solution of the inverse problem which consists in the simultaneous determination of the coefficients $a_{1,2}(u)$ model (1). The results obtained show that observation of the temperature field $u_*(x, t)$ in experiment does not permit a single-valued indication of the specimen thermophysical properties even in the case of giving a normalizing factor. An analogous result was obtained earlier in [1]. But in contrast to this paper, the investigation performed above refines the meaning of nonuniqueness of the solution without at the same time excluding other possible cases of the appearance of ambiguity related, for example, to discrete assignment of the observations. The form of family (3) and the transport group (16) have not been obtained earlier.

Consequences of importance in practice that establish the appearance of nonidentifiable temperature fields for the most typical structural schemes for the experimental determination of the thermophysical properties can be obtained from the properties of the function $u_*(x, t)$. Since the temperature field $u_*(x, t)$ is in the nature of a self-similar solution in variables of form (4) or (5), then satisfaction of the conditions $u_*|_{t=0} \neq \text{const}$ and $\partial u_*/\partial x \neq 0$ is necessary to its existence. Hence, no temperature field $u_*(x, t)$, nonidentifiable in the large, exists in a plate if a constant initial distribution is given or there is thermal insulation of one of its side surfaces. If a boundary condition of the second kind is used in solving the inverse problem, then it also follows from the inequality $\partial u_*/\partial x \neq 0$ that giving a known heat flux on the boundary excludes the appearance of a nonidentifiable temperature field $u_*(x, t)$ in the large.

Let us turn attention to the fact that in this latter case the solution of the heat-conduction equation under consideration is not invariant relative to the extension of its coefficients. In this connection, giving a normalizing factor is not required. And the result obtained displays what the scheme of the experiment can be in order to determine simultaneously the volume specific heat and the heat-conductivity coefficients without utilization of internal bulk heat sources. Let us note that conditions (7) and (3) can also be utilized in planning analogous experiments. The properties mentioned reflect the necessary conditions for the existence of a nonidentifiable temperature field in the large, and can be a criterion for its appearance in the solution of practical problems.

The results obtained, particularly the investigation of (15), permit a further analysis of the properties of self-similar solutions of the heat-conduction equation.

If the coefficients $a_{1,2}(u)$ satisfy the condition

$$\frac{\xi_t}{\xi_x^2}\,a_1 = p\,\frac{da_2}{du}, \tag{17}$$

where p = const, then (15) is homogeneous. Its general solution is expressed by the functions

$$\lambda = \frac{k\dfrac{da_2}{du}}{a_2(a_2 - k)}, \quad k = \text{const}.$$

From the compatibility of conditions (3) and (17) for this case, we obtain p = 1 and k = 0. A self-similar solution of the form $u_* = \xi$ corresponds to these values. For instance, the function $u_* = (a + x)/\sqrt{b(c - t)} + d$ is a solution of the problem (1) if the coefficients $a_1 = 2m\nu \dfrac{u^{m-1}}{u - d}$, $a_2 = \nu u^m$ are given and appropriate boundary conditions are selected.

Let us study the extensively studied case when

$$a_1 = 1,\ a_2 = \nu u^m,\ m \neq 0. \tag{18}$$

For the self-similar variable (4) Eq. (15) is transformed into the form

$$\frac{d\lambda}{du} + \lambda^2 + 2\lambda\frac{m}{u} + \frac{m(m-1)}{u^2} = 0. \tag{19}$$

Its general solution will be the function

$$\lambda = \frac{km + (1 - m)u}{u(u - k)}, \quad k = \text{const}. \tag{20}$$

Then coefficients (18) belong to family (3) for any m and k if $\nu = b/a^2$. Function (20) found is not the unique solution of (19) since the Cauchy problem has not been posed for the latter. A second solution can be obtained by adding the function $\lambda_0 = \dfrac{2m - 1}{ku^{2m} - u}$, $k = \text{const}$, which is a solution of the homogeneous equation (19), to (20). But from the condition of compatibility of (3) and (18), we find that it is necessary to select m = ½ from among the values of m. Then $\lambda_0 = 0$, and hence function (20) is a unique solution of (19), which generates the temperature field u_*. The class of self-similar solutions

$$\int \frac{u_*^m}{u_* - k}\, du_* = ax + bt + c,$$

corresponds to the found value of the parameter λ, from which the solutions obtained in [8] follow.

For the more general case when $a_1 = \nu u^n$, $a_2 = \nu u^m$, the appropriate self-similar solution is described by the quadrature

$$\int \frac{u_*^m}{u_*^{n+1} + k}\, du_* = ax + bt + c,$$

where k is an arbitrary constant. If k = 0, then we obtain a solution of the form $u_* = [(ax + bt + c)(m - n)]^{1/(m-n)}$. Let us note that these same solutions can be obtained for other values of the coefficients $a_{1,2}(u)$ if the transformation (16) is used.

Turning to the self-similar variable (5), we convert (15) to the following form within the framework of coefficients of the type (18) ($m \neq 0, 1$):

$$\frac{d^2\lambda}{du^2} + 4\frac{d\lambda}{du}\lambda + 3\frac{d\lambda}{du}\frac{m}{u} + 2\lambda^3 + 5\lambda^2\frac{m}{u} + 4\lambda\frac{m(m-1)}{u^2} + \frac{m(m-1)(m-2)}{u^3} = 0.$$

The particular solution $\lambda = k/u$ can be indicated here if the constant k satisfies the cubic equation

$$2k^3 - 4k^2 + 5k^2m + (2 - 7m + 4m^2)k + m^3 - 3m^2 + 2m = 0.$$

Only the last root among its roots $k_1 = -m$, $k_2 = -m + 1$, $k_3 = (2 - m)/2$, compatible with conditions (3) and (18), is not violated if d = 0, $\nu = 2/m/(m + 2)$. In this case the self-similar solution $u_* = [m(a + x)/2/\sqrt{b(c - t)}]^{2/m}$ applies. The particular form of such a solution was obtained earlier in [9].

As above, other values of the coefficients of the problem are found by using the transformation (16), and to which the solution found will also correspond. For instance, they might be the following power functions $a_1 = \frac{1}{2}m(2 + 2n - m)\nu u^{n-m} + 1$, $a_2 = \nu u^m + \nu u^n$.

Therefore, the following deductions can be made from the results obtained. Functional properties are found of the solution of a nonlinear heat-conduction equation which is invariant relative to the transport group of its coefficients. Associated with this group is spoilage of the mutually one-to-one correspondence between the coefficients and the solution of the correctly formulated problem. Consequently, the inverse problem governing the thermophysical properties can have a nonunique solution despite the uniqueness of the solution of the direct problem. For practical purposes this means the existence of the possibility of an ambiguous selection of the specimen specific heat and heat-conduction coefficients in the observation of the unique temperature field behind it. An ambiguity subset, for which the given temperature field is not identifiable in the large, is extracted from among the coefficients of the heat-conduction equation. The form of the boundary conditions assuring conservation of the one-to-one mutual correspondence is obtained. Nonidentifiable temperature fields in the large are found for power-law dependences of the thermophysical properties on the temperature. It is shown that the existence of certain known self-similar solutions of the heat-conduction equation is related to the coefficient invariance. Their generalization is performed and new self-similar solutions are found. The results obtained can be recommended for the planning of experiments associated with the complex determination of thermophysical properties.

NOTATION

x, space coordinate; t, time; u(x, t), temperature field; $a_1(u)$, volume specific heat coefficient; $a_2(u)$, heat-conduction coefficient; $\varphi(x)$, $\gamma_{1,2}(t)$, boundary conditions; ρ, λ, parameters of the family; $u_*(x, t)$, nonidentifiable states; and ξ, self-similar variable.

LITERATURE CITED

1. V. V. Frolov, "On the uniqueness of the solution of identification problems of heat-conducting media," Dokl. Akad. Nauk BSSR, 17, No. 1, 25-28 (1973).
2. N. V. Muzylev, "Uniqueness theorems for certain inverse heat-conduction problems," Zh. Vychisl. Mat. Mat. Fiz., 20, No. 2, 387-400 (1980).
3. M. R. Romanovskii, "On uniqueness of the solution of the inverse coefficient problem for the linear heat-conduction equation," Inzh.-Fiz. Zh., 42, No. 3, 476-483 (1982).
4. M. M. Lavrent'ev, V. G. Vasil'ev, and V. G. Romanov, Multidimensional Inverse Problems for Differential Equations [in Russian], Nauka, Novosibirsk (1969).
5. Yu. E. Anikonov, Certain Methods of Investigating Multidimensional Inverse Problems for Differential Equations [in Russian], Nauka, Novosibirsk (1978).
6. O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural'tseva. Linear and Quasilinear Equations of Parabolic Type [in Russian], Nauka, Moscow (1967).
7. L. V. Ovsyannikov, Group Analysis of Differential Equations [in Russian], Nauka, Moscow (1978).
8. G. I. Barenblatt, "On a class of exact solutions of a plane one-dimensional problem of nonstationary gas filtration in a porous medium," Prikl. Mat. Mekh., 17, No. 6, 739-742 (1953).
9. A. A. Samarskii and I. M. Sobol', "Examples of a numerical computation of temperature fields," Zh. Vychisl. Mat. Mat. Fiz., 3, No. 4, 702-719 (1963).